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A Structural Model of Factors Influencing Mathematical Thinking and Reasoning Competence in Vietnamese Upper Secondary Education

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Abstract. Mathematical thinking and reasoning competence is essential in contemporary mathematics education, yet empirical evidence explaining how pedagogical and cognitive factors jointly influence this competence remains limited. This study developed and validated a structural model to examine the relationships among five latent constructs, including: (1) reasoning-oriented task design and learning environment, (2) error-based pedagogical practices, (3) mathematical discourse and communication, (4) students' cognitive disposition and self-regulation, and (5) mathematical thinking and reasoning competence. Data were collected from 246 upper secondary mathematics teachers in Vietnam. Confirmatory factor analysis (CFA) and structural equation modeling (SEM) were used to evaluate the measurement model and examine the proposed relationships. The results showed that error-based pedagogical practices, students' cognitive disposition and self-regulation, and mathematical discourse and communication significantly predicted mathematical thinking and reasoning competence. In addition, reasoning-oriented task design and learning environment positively influenced mathematical discourse and communication, which in turn was significantly associated with students' cognitive disposition and self-regulation. These findings highlight the complementary roles of pedagogical and cognitive factors in developing mathematical thinking and reasoning competence and suggest that the intentional use of students' errors, together with the fostering of mathematical discourse, can support mathematics teaching in upper secondary education.

Keywords: error-based pedagogy; mathematical discourse and communication; mathematical thinking and reasoning competence; self-regulated learning; upper secondary education

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1. Introduction

In the era of AI-assisted learning, computational reasoning competency - the ability to formulate, analyze, and solve problems through logical and computational processes - has become an essential skill for understanding, analyzing, and solving complex real-world problems (Miao & Cukurova, 2024; Wu et al., 2024). Thus, mathematics teaching not only equips students with computational skills but also develops higher-order thinking abilities. Mathematical competence can be considered a multidimensional construct and can be divided into two broad groups (Niss & Højgaard, 2019). The first group comprises mathematical thinking, problem solving, modeling, and reasoning - or internal factors.

The second group focuses on the use of mathematical language and tools (e.g., representation, communication, symbols, and instruments). Within this taxonomy, mathematical thinking and reasoning play a primary role (Ferrini-Mundy, 2000). Niss and Højgaard (2019) emphasized the cognitive processes of these competencies, which are also reflected OECD's PISA framework (OECD, 2017). Conceptually, Tall (2002) defined mathematical thinking as a cognitive process where students take part in analysis, synthesis, abstraction, generalization, inductive and deductive reasoning, and transitions between mathematical representations.

In contrast, mathematical reasoning involves the formulation, articulation, and justification of mathematical statements based on given premises to reach valid conclusions (Mumu & Tanujaya, 2019). These two concepts are not discrete but closely related and together underpin problem solving, reflection, and strategic decision-making in mathematical activity (Polya, 1954; Schoenfeld, 1985). In mathematics, students perform internal cognitive operations and externalize them through language, symbols, and logical structures (Lestari et al., 2022). Due to different cultures, each region has its own set of mathematical competencies. In Vietnam, the 2018 General Mathematics Education Curriculum defines five components: mathematical thinking and reasoning, problem solving, mathematical modeling, mathematical communication, and the use of tools and representations (Bộ Giáo dục và Đào tạo, 2018).

To foster mathematical thinking and reasoning, both pedagogical practices and learners' cognitive characteristics should be carefully considered. Existing studies reported that task design, classroom communication, teacher feedback, the use of student errors, and learners' cognitive dispositions and self-regulation should be taken into account (Jeannotte & Kieran, 2017; Stylianides, 2007; Kleitman & Gibson, 2011). For example, pedagogical activities aimed at stimulating and clarifying students' thought processes through questions, feedback, and discussions can facilitate the development of students' reasoning abilities (Pratiwi et al., 2017; Shaughnessy & Boerst, 2018). Furthermore, error analysis-based teaching has been shown to provide effective learning opportunities in developing proportional reasoning and solving practical problems (Khasawneh et al., 2022). Learner-specific factors (i.e., confidence levels) are positively correlated with mathematical reasoning ability (Kleitman & Gibson, 2011). In this context, Stylianides (2007) described the classroom as an interpretive community where students prove and evaluate reasoning.

In recent years, numerous studies in mathematics education have proposed approaches to improving computational reasoning competency (Begolli et al., 2021; Uyen et al., 2021; Smit et al., 2023; Gunes Uzun, 2024). However, most existing works remain primarily theoretical, qualitatively descriptive, or based on isolated experiments; quantitative evidence regarding the structure of influencing factors and their relationships to mathematical thinking and reasoning abilities remains limited. In particular, studies that integrate classroom pedagogical elements with students' cognitive characteristics within a unified structural model, validated by modern analytical methods such as Confirmatory Factor Analysis (CFA) and Structural Equation Modeling (SEM), have not been systematically implemented in the context of Vietnamese upper secondary education.

To address this gap, this study proposed and validated a theoretical model explaining how pedagogical factors (task design, discourse organization, and error exploitation) and students' cognitive tendencies and self-regulation contribute to mathematical thinking and reasoning. Consequently, there remains limited empirical evidence explaining how these pedagogical and cognitive factors jointly influence mathematical thinking and reasoning competence in Vietnamese upper secondary education. The proposed model was empirically examined using data collected from upper secondary mathematics teachers in Vietnam.

Accordingly, the study sought to answer the following research question: What factors and relationships affect upper secondary students' mathematical thinking and reasoning competence? Specifically, the study aimed to: (1) validate the proposed measurement and structural model using CFA and SEM; (2) examine the direct and indirect effects of pedagogical and cognitive factors on mathematical thinking and reasoning competence; and (3) identify the relative contributions of these factors to provide empirical evidence for improving mathematics teaching practice.

2. Literature Review and Hypothesis Development

In the 2018 General Education Program for Mathematics, mathematical thinking and reasoning competence is identified as one of the core components of mathematical competence. This competency involves performing mathematical thinking operations, applying inductive and deductive reasoning, formulating and answering questions, and explaining or justifying mathematical solutions (Bộ Giáo dục và Đào tạo, 2018). Based on this orientation, the current research approached mathematical thinking and reasoning competence as a latent construct manifested through observed cognitive and reasoning behaviors in classroom practice. This conceptualization is informed by prior studies on mathematical reasoning, classroom discourse, error-based pedagogy, and self-regulated learning (Stylianides, 2007; Kleitman & Gibson, 2011; Khasawneh et al., 2022; Smit et al., 2023). The conceptual model was constructed in a hierarchical manner, where pedagogical factors in the classroom are considered external organizational conditions; students′ cognitive tendencies and self-regulation played the role of individual characteristics; and mathematical thinking and reasoning abilities are the central output. This approach allowed for the simultaneous examination of direct and indirect relationships between structures within an integrated model. The following review synthesized prior research to identify major approaches to the development of mathematical thinking and reasoning and to delineate key groups of influencing factors. Based on the theoretical arguments and empirical evidence reviewed below, the study proposed the following directional hypotheses.

Numerous studies have shown that the design of learning tasks is related to how mathematical discourse and communication are formed in the classroom. Tasks requiring students to explain, justify, or present problem-solving strategies often create opportunities for reasoned interactions, thereby encouraging students to articulate their reasoning and approaches in problem-solving (Darmawijoyo et al., 2025). Studies have also shown that the quality of mathematical communication in the classroom is influenced by the extent to which learning tasks facilitate discussion and justification (Gunes Uzun, 2024; Smit et al., 2023).

Furthermore, tasks that are rich in critical thinking are seen as conditions for students to engage in explanatory, critical, and expansive mathematical discourse during the learning process (Adleff et al., 2023; Siregar et al., 2020; Uyen et al., 2021). Overall, studies show that the design of reasoning-oriented learning tasks is associated with the formation of forms of communication and mathematical discourse in the classroom (Cesaria & Herman, 2019). These results provided evidence for the following hypothesis:

Hypothesis 1 (H1): Reasoning-oriented task design and learning environments have a positive effect on the organization of mathematical discourse and communication in the classroom.

Errors in mathematics learning do not occur randomly, but are often closely linked to the structure, requirements, and design of learning tasks. Santana-Ramírez et al. (2025) showed that tasks can be designed to reveal common misconceptions by requiring explanation and justification, thereby facilitating the clear identification of errors. From a broader perspective, Shimizu and Kang (2025), through a study of student errors in mathematics education from 2018–2023, emphasized that the design of mathematical tasks is a central component in structuring how errors are approached in teaching. Student errors are often identified and addressed in direct relation to learning tasks designed to elicit, clarify, and discuss misunderstandings or misconceptions, rather than appearing as random phenomena outside the context of the task (Shimizu & Kang, 2025).

This approach indicates that the learning task is not only a means of conveying content, but also a fundamental pedagogical condition for errors to become objects of analysis and learning. Borasi (1994) further argued that carefully designed mathematical tasks enable teachers to use students' errors as starting points for inquiry, reflection, and conceptual development rather than treating them merely as incorrect responses. Likewise, Santagata (2005) demonstrated that teachers' mistake-handling practices depend largely on how instructional activities are designed and organized, indicating that well-designed learning tasks create the conditions under which students' errors can be productively analyzed and used to support learning. Together, these studies suggest that reasoning-oriented task design creates favourable pedagogical conditions for implementing error-based pedagogical practices by providing meaningful opportunities for teachers to identify, analyze, and utilize students' errors as valuable learning resources. Based on the empirical evidence and research overview above, the following hypothesis was formed:

Hypothesis 2 (H2): Reasoning-oriented task design and learning environments have a positive effect on error-based pedagogical practices.

Mathematical communication facilitates the emergence and collective analysis of incomplete arguments. Gunes Uzun (2024) showed that argumentative mathematical dialogue helps students clarify weak points. Smit et al. (2023) indicated that discussion supported by formative feedback helps students adjust their reasoning steps. These results provide evidence for the following hypothesis:

Hypothesis 3 (H3): The organization of mathematical discourse and communication has a positive effect on error-based pedagogical practices.

Recent research suggestted that mathematical communication is linked to the formation of students' cognitive traits. Smit et al. (2023) indicated that reasoned dialogue supported by feedback is related to competence beliefs and learning motivation. Gunes Uzun (2024) showed that frequent participation in reasoned dialogue contributes to the formation of students' habits of reasoning, justification, and self-regulation. These results lead to the following hypothesis:

Hypothesis 4 (H4): The organization of mathematical discourse and communication has a positive effect on students' cognitive disposition and self-regulation.

Numerous studies in mathematics education have shown that exploiting and analyzing students' errors is linked to the development of mathematical thinking and reasoning (Gedik Altun et al., 2017; Begolli et al., 2021; Khasawneh et al., 2022). Empirical studies indicate that when errors are incorporated into the learning process as an object of analysis, students must reconsider the structure of the solution, evaluate the reasonableness of the reasoning steps, and adjust their reasoning as needed (Khasawneh et al., 2022; Lin et al., 2025). According to Gedik Altun et al. (2017), error analysis and correction activities organized using a constructivist approach help students develop critical thinking skills in problem-solving.

In an empirical study, Khasawneh et al. (2022) showed that error analysis-based teaching is positively associated with improved students' reasoning abilities, especially when students are required to identify and explain errors in reasoning. Other studies have also shown that analyzing incorrect solutions in mathematical tasks forces students to evaluate the relationship between assumptions, reasoning, and conclusions in the problem-solving process (Begolli et al., 2021).

Recently, Lin et al. (2025) showed that analyzing incorrect solutions generated by AI systems also contributes to improving students' ability to evaluate and adjust reasoning in practical problem-solving situations. Synthesizing these findings suggests that error-based pedagogical activities not only create opportunities for students to identify and correct mistakes, but also promote the evaluation, justification, and revision of reasoning, which are fundamental dimensions of mathematical thinking and reasoning competence. Based on this, the study proposed the following hypothesis:

Hypothesis 5 (H5): Error-based pedagogical practices have a direct effect on students' mathematical thinking and reasoning competence.

In mathematics education, numerous studies have shown that forms of discourse and communication in the classroom are linked to the development of students' mathematical reasoning (Cesaria & Herman, 2019; Siregar et al., 2020; Uyen et al., 2021). Research results indicated that when students engage in reasoned mathematical dialogues, including activities of explanation, justification, and critique, these interactions facilitate students' practice and development of reasoning forms in solving mathematical tasks (Arnesen & Rø, 2022; Smit et al., 2023; Mukuka & Alex, 2024). Research also shows that the frequency and quality of mathematical dialogues, along with the organization of discussions and the provision of feedback in the classroom, are related to the maintenance and development of students' mathematical reasoning discourse (Arnesen & Rø, 2022; Gunes Uzun, 2024; Smit et al., 2023).

Furthermore, empirical studies in various teaching contexts have shown that organizing collaborative interactions and explanatory activities in the classroom increases opportunities for reasoned dialogue among students and is associated with improved mathematical reasoning skills (Adleff et al., 2023; Darmawijoyo et al., 2025; Mukuka & Alex, 2024; Siregar et al., 2020; Uyen et al., 2021). Overall, research results suggest that mathematical communication in the classroom can create a favorable environment for students to practice and develop forms of mathematical reasoning (Cesaria & Herman, 2019). Based on this, the study proposed the following hypothesis:

Hypothesis 6 (H6): The organization of mathematical discourse and communication has a direct effect on students' mathematical thinking and reasoning competence.

Numerous studies have shown that learners' cognitive traits and self-regulatory abilities are related to the quality of reasoning and argumentation outcomes in learning. Empirical studies indicate that self-regulated learning and factors related to competence beliefs are significantly associated with students' reasoning outcomes (Kleitman & Gibson, 2011; Kleitman et al., 2012; Maulida et al., 2024). In mathematics education, studies also show that attitudes toward the subject and self-competence beliefs were directly related to mathematical reasoning ability, and the role of metacognitive processes is considered an important mechanism in this relationship (Leong, et al., 2023; Tak, et al., 2023). Furthermore, studies on cognitive mechanisms show that processes such as reflective abstraction are involved in the formation and operation of mathematical reasoning structures through the coordination of thought operations and control of the reasoning process (Nurrahmah et al., 2023). Overall, these research results suggest that students' cognitive and self-regulatory factors are related to the formation and development of mathematical reasoning. Based on this, the study proposed the following hypothesis:

Hypothesis 7 (H7): Students' cognitive disposition and self-regulation have a direct effect on mathematical thinking and reasoning competence.

Figure 1 presents the conceptual model of the study, constructed based on the proposed hypotheses. In the model, each latent factor is represented by an ellipse, while the hypotheses regarding direct causal relationships between the factors are shown by arrows.

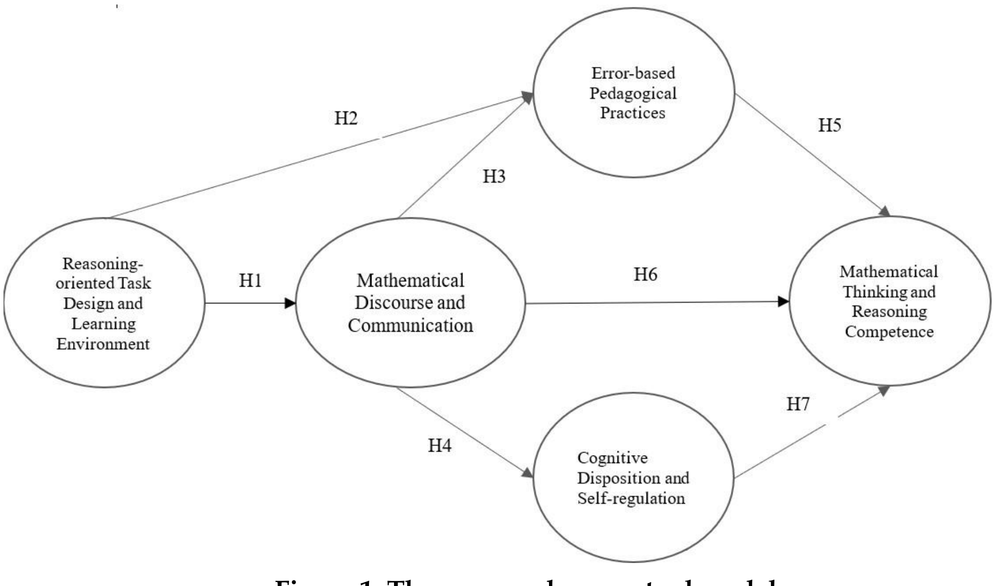

Figure 1: The proposed conceptual model

3. Methodology

3.1 Research Design

This study used a quantitative design, using CFA and SEM to examine the relationship among factors that influence the mathematical thinking and reasoning abilities of secondary school students. The research process consisted of two stages: (1) testing the measurement model to confirm the convergence and discriminant properties of the factors; (2) testing the structural model to estimate the strength and direction of the impact between the factors in the theoretical model.

3.2 Sample and Data Collection

The survey participants were upper secondary mathematics teachers at educational institutions in Vietnam. A non-probability sampling approach was employed, combining convenience sampling and snowball sampling. The survey link was initially distributed to accessible groups of upper secondary mathematics teachers through postgraduate classes, in-service teacher-training courses, professional development workshops, and alumni networks. Participants were also encouraged to forward the survey link to other eligible upper secondary mathematics teachers in their professional networks. Therefore, the sample consisted of teachers who were reachable through the researchers' academic and professional networks and who voluntarily agreed to participate in the study.

Data were collected through an online Google Forms survey. A total of 266 responses were received between February 1, 2026, and February 28, 2026. After removing 20 invalid responses during data cleaning, 246 responses were used for analysis. Regarding sample size, Hair et al. (2019) suggested that the minimum number of observations for a variable should be five to one. In the context of our study, the ratio was 10.1:1, indicating a adequate ratio. Participants were fully informed about the research purpose, voluntariness, and data confidentiality before responding to the survey.

Table 1 presents the characteristics of the 246 survey participants. Regarding gender, the survey sample showed a significant disparity, with female teachers accounting for 60.2%, considerably higher than the 39.8% reported for male teachers. In terms of educational level, the majority of participants held postgraduate degrees, with 55.3% holding Master's degrees and 3.2% holding Doctoral degrees; the Bachelor's degree group accounted for 41.5%. This indicates that the survey participants have a relatively high level of professional expertise.

In terms of years of service, the group of teachers with 5 to 10 years of experience accounted for the largest proportion (41.5%), followed by the group with less than 5 years (26.8%). The combination of young teachers and those with average experience helped to create a more multidimensional data set. Regarding educational locations, teachers came from 13 different provinces and cities across the country, with the highest concentration in Nghe An (27.7%) and Thanh Hoa (14.6%). Including localities from the North, Central, and South of Vietnam increased diversity in the regional context of the survey subjects.

Table 1: Characteristics of the respondents

Category	Group	Frequency	Percentage (%)
Gender	Male	98	39.8
	Female	148	60.2
Teaching Experience	Less than 5 years	66	26.8
	5–10 years	102	41.5
	11–15 years	52	21.1

	More than 15 years	26	10.6
Educational Qualification	Bachelor's degree	102	41.5
	Master's degree	136	55.3
	Doctoral degree	8	3.2
Province/City	Ha Noi	22	8.9
	Son La	18	7.3
	Thai Nguyen	26	10.6
	Thanh Hoa	36	14.6
	Nghe An	68	27.7
	Ha Tinh	26	10.6
	Quang Tri	6	2.4
	Hue	2	0.8
	Dong Nai	16	6.5
	Ho Chi Minh	8	3.3
	Tay Ninh	12	4.9
	Dak Lak	3	1.2
	Lam Dong	3	1.2
Total		246	100

3.3. Measurement and Variable Coding

The study's scale comprised 23 observed variables, distributed among 5 latent factors corresponding to the proposed conceptual model. The questionnaire statements were constructed based on a synthesis and concretization of the theoretical arguments and empirical evidence presented in Section 2. Each group of observed variables reflected a distinct conceptual structure within the model.

The Reasoning-oriented Task Design and Learning Environment consisted of six observed variables (Q1—Q6). This group of variables was based on studies emphasizing the role of learning task design in shaping discourse and the quality of mathematical reasoning in the classroom (Adleff et al., 2023; Darmawijoyo et al., 2025; Gunes Uzun, 2024; Siregar et al., 2020; Smit et al., 2023; Uyen et al., 2021). These studies indicated that tasks that required students to explain, justify, compare solutions, and that were supported by formative feedback were associated with increased student reasoning engagement. Based on this, the variables Q1–Q6 reflected the extent to which teachers design and organize tasks in a way that encourages students to perform higher-order thinking operations, engage in explanatory and justification activities, and emphasize pedagogical feedback and evaluation of the reasoning process in the classroom.

The construct Error-based Pedagogical Practices comprised four observed variables (Q7–Q10). These variables are based on studies that view student errors as a learning resource in mathematics teaching (Begolli et al., 2021; Khasawneh et al., 2022; Santana-Ramírez et al., 2025; Shimizu & Kang, 2025). These studies showed that designing situations that reveal errors and organizing students to analyze, evaluate, and adjust their reasoning can support the development of reasoning. Based on this foundation, the Q7–Q10 variables reflected the extent to which teachers use errors as a pedagogical tool in the classroom, including selecting tasks likely to reveal misconceptions, constructing distractors based on

common errors, and organizing students to analyze, explain, and adjust illogical arguments.
The construct Mathematical Discourse and Communication comprised four observed variables (Q11–Q14). This group of variables was developed based on studies of the role of mathematical discourse and argumentative dialogues in the classroom (Gunes Uzun, 2024; Smit et al., 2023). These studies showed that the way teachers organize discussions that require students to explain, and debate is related to the development of mathematical reasoning. The variables Q11–Q14 therefore reflected the extent to which teachers facilitate students' presentation of ideas, exchange of views, clarification of reasoning steps, and use of mathematical forms of expression during discussions.

The construct Cognitive Disposition and Self-regulation comprised four observed variables (Q15–Q18). This group of variables was based on studies of self-regulated learning, competence beliefs, and metacognitive processes related to reasoning (Leong, et al., 2023; Kleitman & Gibson, 2011; Kleitman et al., 2012; Maulida et al., 2024). These studies showed that the ability to self-monitor reasoning, adjust problem-solving strategies, and the level of perseverance of learners are related to the quality of mathematical reasoning. Based on this, the variables Q15–Q18 reflected manifestations such as perseverance when faced with difficult problems, the ability to self-check the logic of reasoning, adjust strategies when necessary, and reflect on the learning process.

The construct Mathematical Thinking and Reasoning Competence comprised five observed variables (Q19–Q23). These variables were constructed based on the description of mathematical thinking and reasoning components in the 2018 Vietnamese General Mathematics Curriculum,and reference empirical studies on mathematical reasoning in mathematics education (Begolli et al., 2021; Cesaria & Herman, 2019; Gunes Uzun, 2024; Smit et al., 2023; Uyen et al., 2021). Variables Q19–Q23 reflected the core manifestations of this competency, including performing mathematical thinking operations, reasoning and justification, applying inductive and deductive inference, proposing alternative approaches to problem solving, and refining reasoning when necessary. In the research model, this factor was considered the central dependent variable.

All observed variables in the study were measured using a 5-point Likert scale (1 = Strongly Disagree; 5 = Strongly Agree). Table 2 presents a complete list of the questionnaire items used in the study, arranged according to their respective latent factors.

Table 2: Questionnaire items used in the study

No.	Code	Statement
Factor 1. Reasoning-oriented Task Design and Learning Environment (RTDLE)		
1	Q1	Teachers design tasks that require students to explain, justify, or prove rather than merely reproduce knowledge.
2	Q2	Tasks are designed to allow multiple approaches for students to compare and evaluate different solution strategies.
3	Q3	Open-ended questions encourage students to specialize, generalize, or find counterexamples.
4	Q4	Teachers provide specific feedback on the logical coherence and persuasiveness of reasoning.
5	Q5	Assessment emphasizes reasoning processes rather than merely correct or incorrect answers.
6	Q6	The classroom environment encourages students to present ideas supported by reasons and evidence.
Factor 2. Error-based Pedagogical Practices (EPP)		
7	Q7	Teachers select or design tasks that reveal common student misconceptions.
8	Q8	Distractors in multiple-choice questions are constructed based on common errors.
9	Q9	Teachers organize activities for students to analyze and discuss the causes of errors in reasoning.
10	Q10	Teachers guide students to correct errors by restructuring their reasoning.
Factor 3. Mathematical Discourse and Communication (MDC)		
11	Q11	Teachers create opportunities for students to present solutions using both natural language and mathematical language.
12	Q12	Students are encouraged to ask questions and critique their peers' reasoning.
13	Q13	Teachers require students to clarify reasoning steps when presenting solutions.
14	Q14	Students are guided to flexibly use spoken language, written language, and mathematical symbols in reasoning.
Factor 4. Cognitive Disposition and Self-regulation (CDSR)		
15	Q15	Students persist in seeking solutions when facing difficult problems.
16	Q16	Students self-evaluate the logical consistency of their reasoning.
17	Q17	Students adjust their strategies when their approach is ineffective.
18	Q18	Students reflect on how they learn or solve problems.
Factor 5. Mathematical Thinking and Reasoning Competence (MTRC)		
19	Q19	Students perform cognitive operations to identify similarities and differences.
20	Q20	Students explain observed results using appropriate mathematical reasoning.
21	Q21	Students apply inductive and deductive reasoning to construct solution processes.
22	Q22	Students propose alternative approaches and select appropriate solutions.
23	Q23	Students pose and answer questions to clarify reasoning, and can explain, justify, and revise solutions mathematically.

Prior to large-scale administration, the questionnaire underwent a content validation process. All items were derived from the theoretical framework and empirical studies reviewed in Section 2 and were adapted to the context of Vietnamese secondary mathematics education. The preliminary instrument was reviewed by three experts in mathematics education to evaluate content relevance, wording clarity, and construct alignment. Based on their feedback, several items were refined before the final 23-item questionnaire was administered.

3.4. Data Analysis

Data was analyzed using SPSS 26 and AMOS 24. The analysis included descriptive statistics, reliability analysis using Cronbach's alpha, CFA to evaluate the measurement model, and SEM to test the seven proposed hypotheses.

3.5. Model Evaluation Criteria and Hypothesis Testing

The goodness of fit of the CFA–SEM model was assessed based on criteria recommended by Hair et al. (2019), including: CMIN/DF < 3.0; GFI > 0.90; NFI > 0.90; CFI > 0.90; TLI > 0.90; RMSEA < 0.08; PCLOSE > 0.05; and SRMR < 0.08. When these indicators reached the acceptance thresholds, the model was considered to have good fit.

In SEM, hypotheses were accepted when the standardized regression coefficient (β) had a sign consistent with the research hypothesis and a p-value < 0.05, indicating a statistically significant relationship at the 95% confidence level. The reported parameters included the standardized regression coefficient (β), standard error (S.E.), critical value (C.R.), and p-value. In addition, the model's explanatory power for the dependent variable was assessed through the coefficient of determination (R2). According to recommendations in SEM studies, a larger (R2) value indicated a higher explanatory power of the model for the dependent variable; where R2 ≥ 0.50 was considered a strong explanatory power of the model.

4. Results

4.1. Descriptive Statistics

To gain an overview of the data distribution characteristics, the study conducted descriptive statistical analysis of the observed variables in the scale. The indicators used included the mean and standard deviation. The mean reflected the level of agreement among respondents regarding each statement in the scale, while the standard deviation represented the degree of dispersion of the responses. The descriptive statistical results of the 23 observed variables belonging to five factors are presented in Table 3.

Overall, the mean values of the observed variables ranged from 3.11 to 4.04, indicating that most survey participants tended to agree to a moderate to high degree with the statements in the questionnaire. Variable Q9, related to the factor" Error-based Pedagogical Practices," had the highest mean value (Mean = 4.04), reflecting a relatively high level of agreement among respondents regarding the content of this variable. Conversely, variable Q14, related to the factor "Mathematical Discourse and Communication," had the lowest mean value

(Mean = 3.11), indicating a lower level of agreement compared to the other variables in the scale. The standard deviations of the observed variables ranged from 0.86 to 1.02, suggesting a moderate level of data dispersion and that the responses did not exhibit excessive variability. This indicates that the collected data were relatively stable and suitable for further analysis such as scale reliability testing and CFA.

Table 3: Descriptive statistics of observed variables

Construct	Item	Mean	Std. Dev.
RTDLE	Q1	3.25	0.969
	Q2	3.19	0.972
	Q3	3.96	0.958
	Q4	3.43	0.974
	Q5	3.98	0.917
	Q6	3.61	0.961
EPP	Q7	4.00	0.864
	Q8	3.45	1.024
	Q9	4.04	0.925
	Q10	3.64	0.863
MDC	Q11	3.72	1.001
	Q12	3.47	0.942
	Q13	3.50	0.947
	Q14	3.11	0.926
CDSR	Q15	3.28	0.984
	Q16	3.74	0.924
	Q17	4.02	0.923
	Q18	3.95	0.927
MTRC	Q19	3.92	0.991
	Q20	3.72	0.985
	Q21	3.57	0.990
	Q22	3.70	0.997
	Q23	3.37	0.975

4.2. Reliability Analysis and Item–Total Correlation

Before conducting the CFA, the intrinsic reliability of the scales was tested using Cronbach's Alpha coefficient and the adjusted item-total correlation coefficient. According to the recommendation of Hair et al. (2019), a scale was considered reliable when its Cronbach's Alpha coefficient is greater than 0.70, and the observed variables were expected to have an item-total correlation coefficient greater than 0.30. The results of the reliability test of the scales are presented in Table 4. The results showed that all five scales achieved good reliability, with Cronbach's Alpha coefficients ranging from 0.838 to 0.863, exceeding the recommended threshold of Hair et al. (2019). Simultaneously, the item-total correlation coefficients of the observed variables were all greater than 0.50, significantly higher than the minimum threshold of 0.30, indicating that the observed variables were strongly correlated with the overall scale. These findings demonstrate that the measurement scales possess adequate internal consistency to represent the intended latent constructs. Accordingly, all 23 observed variables were retained for the subsequent CFA to evaluate the measurement model.

Table 4: Cronbach's Alpha and corrected item–total correlation of the scales

Construct	Items	Cronbach's Alpha	Item	Corrected Item–Total Correlation
RTDLE	6	0.859	Q1	0.697
			Q2	0.620
			Q3	0.694
			Q4	0.612
			Q5	0.618
			Q6	0.658
EPP	4	0.838	Q7	0.713
			Q8	0.647
			Q9	0.656
			Q10	0.677
MDC	4	0.840	Q11	0.647
			Q12	0.671
			Q13	0.734
			Q14	0.646
CDSR	4	0.858	Q15	0.710
			Q16	0.687
			Q17	0.671
			Q18	0.743
MTRC	5	0.863	Q19	0.701
			Q20	0.565
			Q21	0.767
			Q22	0.704
			Q23	0.682

4.3. Confirmatory Factor Analysis

The CFA was conducted to assess the fit of the measurement model and to test the convergent and discriminant validity between the factors in the research model. The results of the assessment of the model's fit are presented in Table 5.

Table 5: Model fit indices of the CFA model

No.	Fit Index	Value	Threshold	Assessment
1	CMIN/DF	1.364	< 3.0	Good fit
2	GFI	0.905	> 0.90	Good fit
3	NFI	0.899	> 0.90	Acceptable
4	CFI	0.971	> 0.90	Good fit
5	TLI	0.966	> 0.90	Good fit
6	RMSEA	0.039	< 0.08	Good fit
7	PCLOSE	0.965	> 0.05	Good fit
8	SRMR	0.047	< 0.08	Good fit

The results showed that the model's fit indicators all met or exceeded the recommended thresholds in SEM studies. Specifically, CMIN/DF = 1.364 (< 3.0) indicated a very good fit of the model to the data. The GFI = 0.905, CFI = 0.971, and TLI = 0.966 indices were all greater than 0.90, reflecting a good fit of the measurement model. The RMSEA = 0.039 and SRMR = 0.047 indices were both less than 0.08, indicating a low approximation error of the model. At the same time, PCLOSE = 0.965 (> 0.05) indicated that the hypothesis of close fit of the model was not rejected. Although the NFI value (0.899) was marginally below the conventional threshold of 0.90, the difference is negligible. Since the remaining fit indices, including CFI, TLI, RMSEA, SRMR, and PCLOSE, all satisfied the recommended criteria, the measurement model could be considered acceptable based on the overall fit evidence. Overall, these results indicated that the measurement model had a good fit with the survey data.

After confirming the model's fit, the study further examined convergent validity through Composite Reliability (CR) and Average Variance Extracted (AVE). The results in Table 6 show that the CR values of the scales ranged from 0.842 to 0.867, all greater than the threshold of 0.70, while the AVE values ranged from 0.507 to 0.604, all exceeding the threshold of 0.50. This indicates that the observed variables contribute well to the latent factors, confirming the convergent validity of the scales.

Table 6: Composite reliability and average variance extracted

Construct	Number of Items	CR	AVE
RTDLE	6	0.860	0.507
MTRC	5	0.867	0.568
EPP	4	0.842	0.571
CDSR	4	0.859	0.604
MDC	4	0.843	0.573

Finally, the discriminant validity between factors was tested using the Fornell–Larcker criterion, where the square root of the AVE of each factor must be greater than the correlation coefficient between that factor and the other factors. The results in Table 7 show that the square roots of the AVE (values on the main diagonal) were all greater than the corresponding correlation coefficients between the factors. Therefore, it can be concluded that the scales in the model achieved discriminant validity, ensuring relative independence between the research concepts.

Table 7: Fornell–Larcker matrix for assessing discriminant validity

Latent Factor	RTDLE	MTRC	EPP	CDSR	MDC
RTDLE	0.712				
MTRC	0.311	0.754			
EPP	0.158	0.613	0.756		
CDSR	0.428	0.681	0.252	0.777	
MDC	0.573	0.551	0.141	0.668	0.757

In summary, the above results indicate that the measurement model demonstrates good overall fit and satisfactory reliability, convergent validity, and discriminant validity. These findings suggest that the observed variables adequately represent their intended latent constructs and that the proposed measurement model provides a sound basis for evaluating the structural relationships among the constructs in the subsequent SEM analysis.

4.4. Structural Equation Modeling

After the measurement model was validated through CFA, the study then tested the hypothetical relationships between factors using SEM. The results showed that the structural model achieved a good fit with the survey data. Specifically, the model's fit indicators all met the recommended thresholds in SEM research: CMIN/DF = 1.386 (< 3); GFI = 0.902 (> 0.90); CFI = 0.968 (> 0.90); TLI = 0.964 (> 0.90); RMSEA = 0.040 (< 0.08); SRMR = 0.053 (< 0.08); and PCLOSE = 0.951 (> 0.05). Although the NFI value (0.896) was slightly below the conventional threshold of 0.90, this deviation is small. Given that the other major fit indices, including CFI, TLI, RMSEA, SRMR, and PCLOSE, indicate good model fit, the structural model was considered acceptable for hypothesis testing. Overall, these results suggest that the structural model has a good fit with the data and can be used to test research hypotheses.

The results of the hypothesis testing in the structural model are presented in Table 8. The results showed that hypothesis H1 was supported because RTDLE had a positive and statistically significant impact on MDC with a standardized regression coefficient ($\beta = 0.576$, $p < 0.001$). Conversely, hypotheses H2 and H3 were not supported because the effect of RTDLE on EPP ($\beta = 0.078$, $p = 0.413$) and the effect of MDC on EPP ($\beta = 0.121$, $p = 0.210$) were both not statistically significant.

Furthermore, the results showed that MDC had a positive and statistically significant impact on CDSR ($\beta = 0.677$, $p < 0.001$), thus supporting hypothesis H4. Simultaneously, MDC also had a direct impact on MTRC ($\beta = 0.174$, $p = 0.029$), confirming hypothesis H6. In addition, both CDSR and EPP had a positive and statistically significant influence on MTRC, with standardized regression coefficients of $\beta = 0.464$ ($p < 0.001$) and $\beta = 0.494$ ($p < 0.001$), respectively, thus supporting hypotheses H7 and H5.

Table 8: Results of hypothesis testing in the structural model (Note: *** p < 0.001)

Hypothesized Relationship	Code	*β*	S.E.	C.R.	*p*-value	Conclusion
RTDLE → MDC	H1	0.576	0.070	7.254	***	Supported
RTDLE → EPP	H2	0.078	0.087	0.818	0.413	Not supported
MDC → EPP	H3	0.121	0.099	1.254	0.210	Not supported
MDC → CDSR	H4	0.677	0.094	8.389	***	Supported
MDC → MTRC	H6	0.174	0.096	2.189	0.029	Supported
CDSR → MTRC	H7	0.464	0.086	5.624	***	Supported
EPP → MTRC	H5	0.494	0.072	8.078	***	Supported

Figure 2 presents the standardized SEM path diagram of the final structural model, illustrating the standardized path coefficients and the significant and non-significant relationships among the latent constructs.

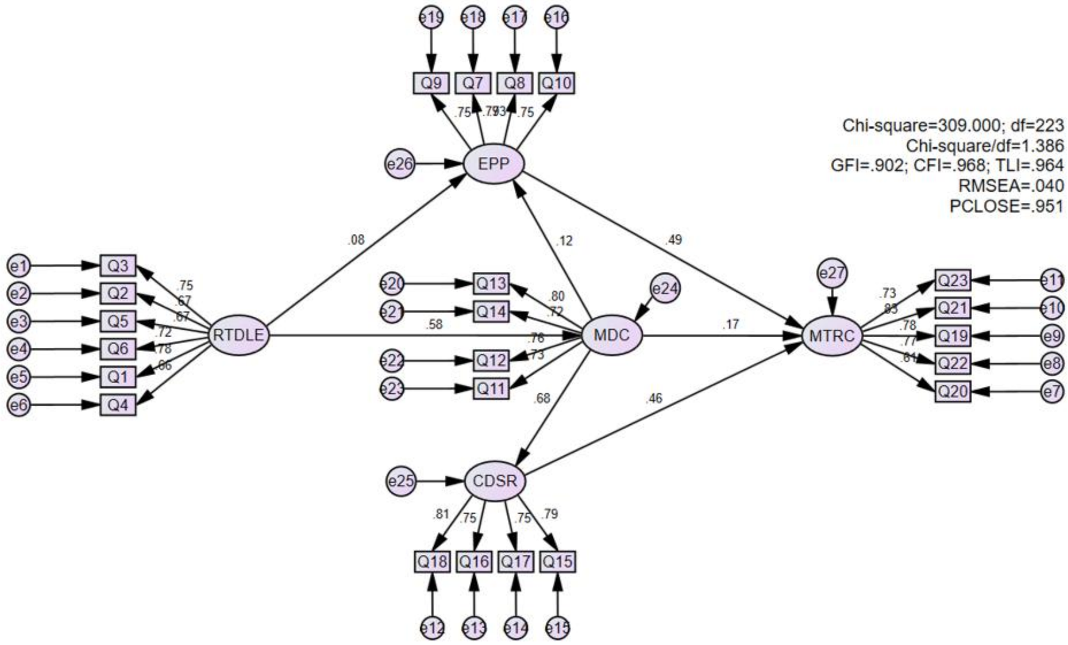

Figure 2: Standardized SEM path diagram of the structural model

Synthesizing the results, 5 out of 7 research hypotheses were accepted, while 2 hypotheses (H2 and H3) were not supported. These results suggest that the organization of mathematical discourse and communication, students' cognitive disposition and self-regulation, and error-based pedagogical practices played a significant role in developing students' mathematical thinking and reasoning abilities in the research model.

The model's explanatory power was assessed through the coefficient of determination (R^2) of the dependent variables in the structural model. The results showed that mathematical discourse and communication had an (R^2) value of 0.331, indicating that RTDLE explained 33.1% of the variation in MDC. For CDSR, the R^2 value was 0.458, meaning MDC explained 45.8% of the variation in CDSR. Notably, MTRC had an R^2 value of 0.679, indicating that the variables MDC, CDSR, and EPP explained 67.9% of the variation in MTRC, reflecting a relatively high level of explanation by the model for students' mathematical thinking and reasoning abilities.

Conversely, EPP had an R^2 value of 0.032, indicating limited explanatory power of the model for this construct. This result implies that teachers' use of error-based pedagogical practices may be affected by additional factors beyond those represented in the current model. Future studies may incorporate these factors to improve the explanatory power of the model with respect to EPP. Overall, the model provides good explanatory power for the mediating variables and especially for the outcome variable MTRC.

5. Discussion

5.1. Validity of the Measurement Model

The CFA results showed that the measurement model achieved a good fit with the survey data. The model's goodness-of-fit indices were all within the recommended thresholds, and the CR and AVE values of the constructs exceeded the acceptable thresholds. This indicated that the scales in the study had appropriate reliability and convergent validity. Furthermore, the Fornell–Larker discriminant validity test results showed that the latent constructs in the model could be distinguished relatively clearly conceptually. These results are consistent with the recommendations in SEM methodology research that scales should simultaneously achieve reliability, convergent validity, and discriminant validity before proceeding with structural model analysis (Hair et al., 2019). In this context of research, the CFA results showed that the five latent constructs were measured with relative stability through the observed variables.

Furthermore, the construct of error-based pedagogical practices also met the criteria for reliability and convergent validity in the measurement model, indicating that this scale is consistently measured within the research model. This result allowed this construct to be used in subsequent structural model analyses. In mathematics education studies, error exploitation is often seen as a deliberate pedagogical practice aimed at promoting students' explanatory, analytical, and reasoning-correcting activities (Begolli et al., 2021; Khasawneh et al., 2022).

5.2. Discussion of the Structural Model

The SEM results showed that five hypotheses in the model were supported by the data. Among them, RTDLE had a positive influence on MDC. This result is consistent with previous studies suggesting that the characteristics of learning tasks can shape the form of interaction in the mathematics classroom. Tasks requiring students to explain, justify, or compare problem-solving strategies often create more opportunities for in-depth mathematical discussion (Darmawijoyo et al., 2025; Gunes Uzun, 2024; Smit et al., 2023). The present findings further indicate that reasoning-oriented task design not only facilitates students' engagement with mathematical content but also helps establish a classroom environment in which communication becomes an integral part of mathematical reasoning.

The results also showed that MDC had a strong positive effect on CDSR. This is consistent with studies suggesting that engaging in structured mathematical dialogues helps students develop habits of explaining, critically analyzing, and evaluating the rationality of reasoning (Gunes Uzun, 2024; Smit et al., 2023). Through explaining, questioning, and evaluating alternative solutions, students are encouraged to monitor and refine their own reasoning. Such dialogic interactions promote cognitive self-regulation by supporting self-monitoring and reflective evaluation during mathematical learning.

Furthermore, MDC also had a direct positive effect on MTRC. This result is consistent with studies showing that explanatory, justification, and critical thinking activities in the classroom can facilitate students' practice of mathematical reasoning (Adleff et al., 2023; Arnesen & Rø, 2022). Although its direct effect is smaller than those of EPP and CDSR, the findings indicated that mathematical discourse provides an important pedagogical context in which students articulate, justify, and critically evaluate mathematical ideas, thereby supporting the development of mathematical thinking and reasoning competence.

The two factors with the strongest direct influence on MTRC were EPP and CDSR. This result is consistent with previous studies showing that error analysis and correction can promote key cognitive processes such as explanation, comparison, and justification (Begolli et al., 2021; Gedik Altun et al., 2017; Khasawneh et al., 2022). Kleitman and Gibson (2011) and Kleitman et al. (2012) indicated that error exploitation in mathematics teaching can support core cognitive operations of mathematical thinking and reasoning. The results of the current study further indicated that EPP is a strong direct predictor of MTRC in the model. This finding suggests that students benefit not merely from identifying errors but from instructional practices that encourage explanation, reflection, and the revision of reasoning. Such practices transform students' errors into meaningful learning opportunities, thereby strengthening mathematical thinking and reasoning.

5.3. Interpretation of Unsupported Hypotheses

Hypotheses H2 (RTDLE → EPP) and H3 (MDC → EPP) were not supported by the data. This result suggests that error-based pedagogical practices are not directly determined by reasoning-oriented task design or mathematical discourse alone. Rather, although these pedagogical conditions may create opportunities for students' errors to emerge and become visible, the instructional use of such errors depends on teachers' deliberate pedagogical decisions.

This interpretation is consistent with Borasi (1994), who argued that students' errors become valuable learning opportunities only when teachers intentionally transform them into starting points for mathematical inquiry rather than treating them merely as incorrect responses. This interpretation is further supported by Santana-Ramírez et al. (2025) and Shimizu and Kang (2025), who emphasized that reasoning-oriented tasks facilitate the identification of misconceptions by encouraging explanation and justification, but do not by themselves ensure that errors will be systematically exploited during instruction.

A similar interpretation applies to H3. Although mathematical discourse provides opportunities for students to explain, justify, and compare mathematical ideas, the pedagogical use of students' errors requires teachers to purposefully organize classroom interactions that encourage students to analyze, discuss, and learn from those errors. Santagata (2005) showed that mistake-handling is shaped by teachers' instructional practices and beliefs rather than arising automatically from classroom interaction itself. Similarly, Stylianides (2007) emphasized the teacher's central role in cultivating mathematical reasoning by orchestrating classroom practices that promote justification and proof, indicating that productive classroom discourse requires deliberate pedagogical guidance.

The unsupported H2 and H3 should therefore not be interpreted as contradicting previous research. Instead, they suggest that reasoning-oriented task design and mathematical discourse function primarily as enabling pedagogical conditions, whereas error-based pedagogical practices represent a distinct instructional process requiring teachers' intentional decisions about when and how students' errors are analyzed, discussed, and transformed into learning opportunities. Consequently, the findings imply that additional teacher-related instructional factors and contextual influences beyond those represented in the present model may contribute to explaining variation in EPP.

5.4. Contributions and Practical Implications

The research model explained approximately 67.9% of the variation in MTRC, indicating a relatively high level of explanation for the central outcome variable. This result provided empirical evidence for an integrated approach in mathematics education research, where competency was considered a result of the interaction between classroom pedagogical factors and students' cognitive characteristics. Compared to previous studies that often considered each factor individually, this study showed that these factors can be linked in a hierarchical influence structure: RTDLE influences MDC; MDC influences CDSR; while CDSR and EPP are direct predictors of MTRC.

The research findings also contributed to bridging two research directions in mathematics education: research on mathematical discourse in the classroom and research on exploiting errors as a learning resource. Previous studies have shown that mathematical discourse plays an important role in promoting students' explanatory and justification processes (Gunes Uzun, 2024; Smit et al., 2023), while other studies emphasize the value of error analysis as a learning opportunity (Nguyễn Văn Thuận et al., 2026). In this study's model, mathematical discourse played the role of creating an environment for reasoning, while error analysis and error correction were more directly related to the development of cognitive abilities.

From a practical perspective, the findings suggest that developing students' mathematical thinking and reasoning competence requires an integrated instructional approach. Teachers should design reasoning-oriented tasks that encourage explanation and justification, organize classroom discourse to support students' reflection on mathematical ideas, and deliberately use students' errors as opportunities for discussion and conceptual refinement. In addition, fostering students' cognitive self-regulation through planning, monitoring, and evaluating their own reasoning may further strengthen mathematical thinking and reasoning competence.

5.5. Limitations and Future Research

This study has several limitations that should be considered when interpreting the findings. First, the model was developed using teacher-reported survey data. Therefore, the findings reflected teachers' perceptions of classroom practices rather than direct measures of students' mathematical thinking and reasoning competence. Although teachers provide valuable professional insights into classroom practice, their responses may also be influenced by respondent subjectivity and common method bias. Second, the cross-sectional design allows the identification of empirical associations but does not support causal interpretations. In addition, participants were recruited using convenience sampling and snowball sampling. Although teachers from multiple provinces in

Vietnam participated in the study, the use of nonprobability sampling may limit the generalizability of the findings.

Future research could combine teacher surveys with classroom observations, student performance assessments, longitudinal studies, or experimental designs to provide stronger evidence for the mechanisms proposed in this model. Future studies could also examine additional teacher-related instructional factors and contextual variables that may better explain error-based pedagogical practices. Finally, future research may investigate how AI assisted learning environments influence the relationships among classroom pedagogy, cognitive self-regulation, and mathematical thinking and reasoning competence.

6. Conclusion

This study developed and validated a model to explain the factors influencing the mathematical thinking and reasoning competence of upper secondary students. The results showed that this competence was simultaneously influenced by pedagogical factors in the classroom and the cognitive characteristics of the students. Specifically, pedagogical practices based on student errors, cognitive self-regulatedtendencies, and mathematical communication all had significant direct influences, with error exploitation and self-regulation playing a prominent role. Simultaneously, the results also indicated that error exploitation does not arise naturally from tasks or communication but is linked to the teacher's deliberate pedagogical strategies.

These findings contribute to clarifying the influence structure in mathematics teaching, showing that the development of mathematical thinking and reasoning competence depends not only on task design but also on how interactions are organized and learning situations are handled in the classroom. In practical terms, the results suggest that teachers should focus on designing tasks that encourage explanation and justification, organizing mathematical communication, supporting students' self-correction, and exploiting errors as learning opportunities. The findings also highlight the importance of fostering students' cognitive self-regulation as an integral part of mathematics teaching.

This study is limited by its cross-sectional design and the use of teacher-reported data. Therefore, the findings should be interpreted as empirical associations rather than causal relationships. Further studies could expand the model using classroom observation data, longitudinal or experimental designs, while also considering the development of this competence in different educational contexts and overtime. As AI-assisted learning environments continue to evolve, future research may further examine how they influence the development of mathematical thinking and reasoning competence.

Conflict of Interest
The authors declare no conflict of interest.

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